

# Irreducibility of $G$ -varieties defined by quadrics.

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## Abstract

Let  $\mathfrak{g}$  be a complex simple Lie algebra,  $G$  a simply connected Lie group with Lie algebra  $\mathfrak{g}$  and  $V$  a module. We will study the irreducibility of  $G$ -varieties defined by quadrics in  $\mathbb{P}V$ .

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## Introduction.

We will study a question raised in the exercise [4, 15.44 Hard Exercise]. Let  $V = \mathbb{C}^n$  be the standard representation of  $\mathfrak{sl}_n(\mathbb{C})$  and consider the following decomposition

$$S^2(\wedge^k V) = \bigoplus_{i \geq 0} \Theta_{2i},$$

where  $\Theta_{2i}$  is an irreducible representation of  $\mathfrak{sl}_n(\mathbb{C})$ . Let  $C^p(Gr^k(V))$  be the  $p$ -restricted chordal variety of the Grassmannian of subspaces of dimension  $n - k$ : that is, the union of chords  $\overline{LM}$  joining pair of planes meeting in a subspace of dimension at least  $k - 2p + 1$ . In the exercise we must prove that the ideal in degree two of  $C^p(Gr^k(V))$  is

$$I(C^p(Gr^k(V)))_2 = \bigoplus_{i \geq p} \Theta_{2i},$$

and the authors asked what is the actual zero locus of these quadrics?. In the present paper we will generalize the situation to the following:

Let  $\mathfrak{g}$  be a simple Lie algebra and let  $G$  be the simply connected Lie group with Lie algebra  $\mathfrak{g}$ , let  $V$  be a representation and  $\overline{G.y} \subseteq \mathbb{P}V$  be the closure of an orbit in the projective space  $\mathbb{P}V$ .

**Theorem.** *The zero locus of quadrics in  $I(G.y)_2$  is an irreducible variety.*

As an application of this result, we will prove that there exist  $y \in C^p(Gr^k(V))$  such that  $I(C^p(Gr^k(V)))_2 = I(G.y)_2$  and then, the zero locus of  $I(C^p(Gr^k(V)))_2$  is an irreducible variety. This gives an answer to the question in [4, 15.44 Hard Exercise].

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After the previous motivation let's present our notations. We will work with a simple Lie algebra  $\mathfrak{g}$ , a module  $V$  and a simply connected Lie group  $G$  with Lie algebra  $\mathfrak{g}$ . For each  $y \in \mathbb{P}V$  consider the orbit  $G.y \subseteq \mathbb{P}V$  and the zero locus of its ideal in degree two,

$$M_y = \{x \in \mathbb{P}V \mid q(x) = 0 \forall q \in I(G.y)_2\}.$$

We will prove that  $M_y$  is irreducible. If the closure of the orbit  $\overline{G.y}$  is defined by quadrics, the variety  $M_y$  is obviously irreducible. It is equal to  $\overline{G.y}$ . Also, if the vector  $y$  corresponds to a maximal weight vector of  $V$ , then the orbit is automatically closed, [4, p.388, claim 23.52], for example, the Veronese variety, the Grassmannian and partial flag varieties [3, §9.3]. In these cases, the variety  $M_y$  is irreducible by trivial reasons (it is an orbit).

This article is divided in four sections. In **section one** we give some preliminaries. In **section two** we define the notion of a multi-matrix. The space of multi-matrices arise naturally in the proof of the irreducibility of  $M_y$ . In **section three** we prove that the variety  $M_y$  is irreducible (see 11). First we show in 6 that for every  $y \in V$  there exist a multi-matrix  $A$  such that

$$M_y \cong \{ABA^t \mid B \in \text{Cat}, \text{rk}(ABA^t) \leq 1\},$$

where  $\text{Cat}$  is the space of catalectic multi-matrices (see definitions in 2). Second, in 9, we give a characterization of the space  $\{ABA^t \mid B \in \text{Cat}\}$ , and with this, we prove in 11 that  $M_y$  is isomorphic to the irreducible variety  $\{P^2 \mid P \in \text{im} A^t\}$  where  $P^2$  is the square of the polynomial  $P$ . In **section four** we give some applications of the result.

## 1. Preliminaries.

We will work with the universal enveloping algebra of  $\mathfrak{g}$ ,

$$U\mathfrak{g} = \left( \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n} \right) / I, \quad I = \langle D \otimes E - E \otimes D - [D, E] \rangle.$$

Elements of  $U\mathfrak{g}$  are the classes of non-commutative polynomials in  $\mathfrak{g}$ . Our goal in this section is to prove the following

**Theorem.** *Let  $y \in V$ , then there exists  $r \in \mathbb{N}$ ,  $(D_1, \dots, D_r) \in \mathfrak{g}^r$ ,  $N \in \mathbb{N}_0^r$  such that for every  $Q \in U\mathfrak{g}$ , we have*

$$Q(yy) = \sum_{i_1, j_1=0}^{N_1} \dots \sum_{i_r, j_r=0}^{N_r} b_{i_1+j_1, \dots, i_r+j_r} \frac{D_1^{i_1} \dots D_r^{i_r} y}{i_1! \dots i_r!} \frac{D_1^{j_1} \dots D_r^{j_r} y}{j_1! \dots j_r!},$$

where the coefficients  $b_{0, \dots, 0, \dots, b_{2N_1, \dots, 2N_r}}$  depends on  $Q$ . In a more compact form, the formula is

$$Q(yy) = \sum_{i, j=0}^N b_{i+j} \frac{D^i y}{i!} \frac{D^j y}{j!}.$$

This theorem will help us to study the irreducibility of  $M_y$ . But first let's start with the following Lemmas.

**Lemma 1.** Let  $D = (D_1, \dots, D_r) \in \mathfrak{g}^r$  and  $n \in \mathbb{N}_0^r$  then

$$\frac{D_1^{n_1} \dots D_r^{n_r}(yy)}{n_1! \dots n_r!} = \sum_{i_1+j_1=n_1, \dots, i_r+j_r=n_r} \frac{D_1^{i_1} \dots D_r^{i_r} y}{i_1! \dots i_r!} \frac{D_1^{j_1} \dots D_r^{j_r} y}{j_1! \dots j_r!}.$$

In a more compact notation we have,

$$\frac{D^n(yy)}{n!} = \sum_{i+j=n} \frac{D^i y}{i!} \frac{D^j y}{j!}$$

where  $D^k := D_1^{k_1} \dots D_r^{k_r}$  and  $k! := k_1! k_2! \dots k_r!$ .

PROOF. Given  $D_r \in \mathfrak{g}$ , we have

$$D_r^k(ab) = \sum_{l=0}^k \binom{k}{l} (D_r^l a)(D_r^{k-l} b) \iff \frac{D_r^k(ab)}{k!} = \sum_{p+q=k} \frac{D_r^p a}{p!} \frac{D_r^q b}{q!}.$$

The result follows by induction.  $\square$

We are assuming that the Lie algebra  $\mathfrak{g}$  is simple. For each positive root  $\beta$ , let  $X_\beta \in \mathfrak{g}^\beta$ ,  $Y_\beta \in \mathfrak{g}^{-\beta}$  and  $H_\beta \in \mathfrak{h}$  such that  $[X_\beta, Y_\beta] = H_\beta$ . From a result in [7, p.57] we know that if  $W$  is irreducible and has a maximal weight vector  $w$ , then  $Y_{\beta_1}^{m_1} \dots Y_{\beta_k}^{m_k} w$ ,  $m_i \in \mathbb{N}_0$ , generates  $W$  as a vector space. In the next Lemma we will prove that there is a similar result without the hypothesis on  $w$ .

**Lemma 2.** Let  $W$  be a finite dimensional representation. Given  $w \in W$  there exists  $r \in \mathbb{N}$ ,  $(D_1, \dots, D_r) \in \mathfrak{g}^r$  with

$$D_i \in \{X_{\beta_1}, Y_{\beta_1}, \dots, X_{\beta_k}, Y_{\beta_k}\},$$

such that  $\{D_1^{m_1} \dots D_r^{m_r} w\}_{m_1, \dots, m_r \geq 0}$  generates  $U\mathfrak{g}w$  as a vector space. In a more compact form, the set may be written as  $\{D^m w\}_{m \geq 0}$ .

PROOF. Let  $p_1, \dots, p_s \in U\mathfrak{g}w$  be the maximal weight vectors of the representation  $U\mathfrak{g}w$  and let  $P_1, \dots, P_s \in U\mathfrak{g}$  be the non-commutative polynomials such that  $P_i w = p_i$ . By the Poincar-Birkhoff-Witt Theorem, [4, p.486], if  $E_1, \dots, E_s$  is a basis of  $\mathfrak{g}$  then every element in  $U\mathfrak{g}$  is a linear combination of the monomials

$$\{E_1^{m_1} \dots E_s^{m_s}\}, \quad m_1, \dots, m_s \geq 0.$$

We will consider the basis of  $\mathfrak{g}$  obtained by the root decomposition,

$$\{X_{\beta_1}, Y_{\beta_1}, H_{\beta_1}, \dots, X_{\beta_k}, Y_{\beta_k}, H_{\beta_k}\}.$$

Given that  $H = [X, Y] = XY - YX$  in  $U\mathfrak{g}$  we may suppose that  $H$  does not appear in the monomials of  $P_i$ . Let  $D_1 = Y_{\beta_1}, \dots, D_k = Y_{\beta_k}$  and let's define  $D_{k+1}, \dots, D_r$ . Let  $D_{k+1}$  be the first variable of the first monomial of  $P_1$ ,  $D_{k+2}$  the second variable of the first monomial of  $P_1$ , finally let  $D_r$  be the last variable of the last monomial of  $P_s$ .

Note that with all the polynomials formed with the monomials of the form  $D_1^{m_1} \dots D_r^{m_r} w$ , we obtain, in particular, the polynomials

$$Y_{\beta_1}^{m_1} \dots Y_{\beta_k}^{m_k} P_i w = Y_{\beta_1}^{m_1} \dots Y_{\beta_k}^{m_k} p_i$$

that generates, as a vector space, the whole representation  $U\mathfrak{g}p_i$ .  $\square$

**Lemma 3.** Let  $V$  be a finite dimensional representation,  $r \in \mathbb{N}$  and  $(D_1, \dots, D_r) \in \mathfrak{g}^r$ ,

$$D_i \in \{X_{\beta_1}, Y_{\beta_1}, \dots, X_{\beta_k}, Y_{\beta_k}\}.$$

Given  $u \in V$  there exist  $N \in \mathbb{N}_0^r$  such that

$$D_1^{N_1+k_1} D_2^{N_2+k_2} \dots D_r^{N_r+k_r} u = 0, \quad \forall k_1, \dots, k_r \geq 0.$$

In a more compact form, we may write

$$D^{N+k} u = 0, \quad \forall k \geq 0.$$

PROOF. Assume first that  $u$  has a particular weight  $\mu$ , that is,  $u \in V^\mu$ . If  $D_r = X_\beta$  then  $D_r^m u \in V^{\mu+m\beta}$ , else if  $D_r = Y_\beta$  then  $D_r^m u \in V^{\mu-m\beta}$ . Given that  $V$  is finite dimensional it has finite weights, then there exist  $\ell \in \mathbb{N}$  such that  $D_r^\ell u = 0$ .

Assume now that  $u$  is a general vector of  $V$ , so we can decompose it as  $u = \sum u_i$  where each  $u_i$  has weight  $\mu_i$ . From the previous paragraph we know that for each  $i$  there exist  $\ell_i$  such that  $D_r^{\ell_i} u_i = 0$ . So if we take the maximum of  $\{\ell_i\}$ , there exist  $\ell \in \mathbb{N}$  such that  $D_r^\ell u = 0$ .

Finally, let's see that for a given  $u \in V$ , there exist  $(N_1, \dots, N_r) \in \mathbb{N}_0^r$  such that

$$D_1^{N_1+k_1} D_2^{N_2+k_2} \dots D_r^{N_r+k_r} u = 0, \quad \forall k_1, \dots, k_r \geq 0.$$

Let  $N_r$  be such that  $D_r^{N_r} u = 0$ . Let  $N_{r-1}$  be the maximum of  $\{\ell_i\}$  where  $\ell_i$  is such that  $D_{r-1}^{\ell_i} (D_r^{N_r} u) = 0$  for  $0 \leq i \leq N_r$ . In general, let  $N_s$  be such that  $D_s^{N_s} (D_{s+1}^{N_{s+1}} \dots D_r^{N_r} u) = 0$  for all  $0 \leq i_j \leq N_j$ .  $\square$

**Theorem 4.** Let  $y \in V$ , then there exists  $r \in \mathbb{N}$ ,  $(D_1, \dots, D_r) \in \mathfrak{g}^r$ ,  $N \in \mathbb{N}_0^r$  such that for every  $Q \in U\mathfrak{g}$ , we have

$$Q(y) = \sum_{i,j=0}^N b_{i+j} \frac{D^i y}{i!} \frac{D^j y}{j!}.$$

where the coefficients  $b_0, \dots, b_{2N}$  depends on  $Q$ .

PROOF. From 2 there exists  $(D_1, \dots, D_r) \in \mathfrak{g}^r$  such that  $\{D^n(y)\}_{n \geq 0}$  generates  $U\mathfrak{g}(y)$  as a vector space. From 3 there exist  $N$  big enough such that  $\{D^n(y)\}_{n=0}^{2N}$  still generates  $U\mathfrak{g}(y)$  and also  $D^{N+k} y = 0$  for  $k \geq 0$ . Finally,

$$Q(y) = \sum_{n=0}^{2N} b_n \frac{D^n(y)}{n!} = \sum_{n=0}^{2N} \sum_{i+j=n} b_n \frac{D^i y}{i!} \frac{D^j y}{j!} = \sum_{i,j=0}^N b_{i+j} \frac{D^i y}{i!} \frac{D^j y}{j!}.$$

The first equality follows because  $\{D^n(y)\}_{n=0}^{2N}$  generates  $U\mathfrak{g}(y)$  as a vector space, the second equality follows from 1 and the last equality follows from the fact that  $D^{N+k} y = 0$  for every  $k \geq 0$ .  $\square$

## 2. Multi-matrixes.

The definitions of multi-matrix and multi-vector given here are a particular case of the definition of matrix in [1].

Let  $r \in \mathbb{N}$ , for each  $N = (N_1, \dots, N_r) \in \mathbb{N}_0^r$  let

$$\underline{N} := \{(i_1, \dots, i_r) \mid 0 \leq i_k \leq N_k\}.$$

A *multi-vector* is a function  $v : \underline{N} \rightarrow \mathbb{C}$ , equivalently, an element of  $\mathbb{C}^{\underline{N}}$ . A *multi-matrix* is an element  $A \in \mathbb{C}^{\underline{N}' \times \underline{N}}$ .

For each  $i, j \in \mathbb{N}_0^r$ , let  $A_{ij} := A(i, j)$  be the *coordinates* of the multi-matrix  $A$ .

In the vector space of multi-matrixes we have operations of *addition*, *product* and *transpose*. The addition is defined if the multi-matrixes are of the same size. The product  $AA'$  is defined if  $A \in \mathbb{C}^{\underline{N}' \times \underline{N}}$ ,  $A' \in \mathbb{C}^{\underline{N} \times \underline{N}''}$ .

$$(A + A')_{ij} = A_{ij} + A'_{ij}, \quad (AA')_{ij} = \sum_{k \geq 0}^N A_{ik} A'_{kj}, \quad (A^t)_{ij} = A_{ji}$$

The notation  $\sum_{k \geq 0}^N$  means  $\sum_{k \in \underline{N}}$ .

A multi-matrix  $B \in \mathbb{C}^{\underline{N} \times \underline{N}}$  is *catalectic* if  $B_{ij} = b_{i+j}$  for some  $b \in \mathbb{C}^{\underline{2N}}$ . The projective space of catalectic multi-matrixes is

$$Cat := \{\langle B \rangle \in \mathbb{C}^{\underline{N} \times \underline{N}} \mid B_{ij} = b_{i+j}, b \in \mathbb{C}^{\underline{2N}}\}.$$

Note that a catalectic multi-matrix  $B$  is symmetric,  $B^t = B$ .

## 3. The irreducibility of $M_y$ .

Let  $G$  be a simple Lie group with Lie algebra  $\mathfrak{g}$ , let  $V$  be a finite dimensional representation and let  $y \in V$  be a non-zero vector. Recall the definition of  $M_y$ ,

$$M_y = \{x \in \mathbb{P}V \mid q(x) = 0 \forall q \in I(G.y)_2\}.$$

**Lemma 5.** *The variety  $M_y$  may be defined as*

$$M_y = \{\langle x \rangle \in \mathbb{P}V \mid xx \in U\mathfrak{g}(yy)\}.$$

where  $U\mathfrak{g}(yy)$  is the smallest  $\mathfrak{g}$ -module that contains  $yy \in S^2(V)$ .

**PROOF.** Consider the vector space generated by the elements of the form  $g.yy \in S^2(V)$ ,

$$S = \langle g.yy \mid g \in G \rangle \subseteq S^2(V).$$

The vector space  $S$  is the smallest  $G$ -module that contains  $yy$ . Using the  $G$ -isomorphism  $\phi : S^2(V^\vee) \rightarrow S^2(V)^\vee$  we can identify a quadratic polynomial  $q \in I(G.y)_2$  with a linear functional  $\phi_q$  such that  $\phi_q(xx) = 2q(x)$ . In fact we have the following,

$$S^\circ := \{\phi \in S^2(V)^\vee \mid \phi(s) = 0 \forall s \in S\} =$$

$$\begin{aligned} \{\phi \in S^2(V)^\vee \mid \phi(gy, gy) = 0 \forall g \in G\} &\cong \\ \{q \in S^2(V^\vee) \mid q(gy) = 0 \forall g \in G\} &= I(G.y)_2. \end{aligned}$$

Given that  $S$  is the smallest  $G$ -module that contains  $yy$ , it is equal to the  $\mathfrak{g}$ -module  $U\mathfrak{g}(yy)$ , then

$$\begin{aligned} M_y &= \{x \in \mathbb{P}V \mid q(x) = 0 \forall q \in I(G.y)_2\} = \{x \in \mathbb{P}V \mid \phi_q(xx) = 0 \forall q \in I(G.y)_2\} = \\ &= \{x \in \mathbb{P}V \mid \phi(xx) = 0 \forall \phi \in S^\circ\} = \{x \in \mathbb{P}V \mid xx \in S\} = \{x \in \mathbb{P}V \mid xx \in U\mathfrak{g}(yy)\}. \end{aligned}$$

□

**Theorem 6.** *Let  $y \in V$  and  $\ell = \dim V$ , then there exists a multi-matrix  $A \in \mathbb{C}^{\ell \times N}$  depending on  $y$  such that*

$$M_y \cong \phi_A(\text{Cat}) \cap \mathcal{V}$$

where  $\phi_A(B) = ABA^t$  and  $\mathcal{V}$  is the Veronese variety,  $\mathcal{V} = \{\langle xx^t \rangle \mid x \in \mathbb{C}^{\ell \times 1}\}$ . Note that the space  $\phi_A(\text{Cat}) \cap \mathcal{V}$  consist of symmetric  $\ell \times \ell$ -matrixes, i.e.  $\phi_A(\text{Cat}) \cap \mathcal{V} \subseteq \mathbb{P}S^2(\mathbb{C}^\ell)$ .

PROOF. Consider the Veronese map  $v_2$ ,

$$v_2 : \mathbb{P}V \longrightarrow \mathbb{P}S^2(V), \quad \langle x \rangle \longrightarrow \langle xx \rangle.$$

Its image is the Veronese variety  $\mathcal{V}$ . From [5, exercise 2.8] we know that  $M_y \cong v_2(M_y)$ ,

$$v_2(M_y) = \{\langle xx \rangle \mid \langle x \rangle \in M_y\} = \{\langle xx \rangle \mid xx \in U\mathfrak{g}(yy)\} = \{\langle xx \rangle \mid xx = Q(yy), Q \in U\mathfrak{g}\}.$$

Fix a basis of  $V$ ,  $\{v_k\}_1^\ell$ , then we can write the elements  $D^i y / i!$ ,

$$\frac{D^i y}{i!} = \sum_{k=1}^\ell a_{ik} v_k,$$

then, by 4 we have

$$Q(yy) = \sum_{i,j=0}^N b_{i+j} \frac{D^i y}{i!} \frac{D^j y}{j!} = \sum_{k,l=1}^\ell \left( \sum_{i,j=0}^N b_{i+j} a_{ik} a_{jl} \right) v_k v_l.$$

The element  $Q(yy)$  is of the form  $xx$ , where  $x = \sum_{k=1}^\ell \lambda_k v_k$  if and only if,

$$\left( \sum_{k=1}^\ell \lambda_k v_k \right) \left( \sum_{l=1}^\ell \lambda_l v_l \right) = Q(yy) \iff \sum_{i,j=0}^N b_{i+j} a_{ik} a_{jl} = \lambda_k \lambda_l, \quad \forall k, l \iff \langle ABA^t \rangle \in \mathcal{V}.$$

where  $B \in \mathbb{C}^{N \times N}$  and  $A \in \mathbb{C}^{\ell \times N}$  are such that  $B_{ij} = b_{i+j}$ ,  $A_{ki} = a_{ik}$ . □

With this theorem at hand, we can now prove the irreducibility of  $M_y$ . First we will characterize  $\phi_A(\text{Cat})$  and then its intersection with  $\mathcal{V}$ . Let's introduce some notations,

**Notation 7.** Let  $V_1, V_2 \subseteq V$  be two linear subspaces, then we will denote

$$V_1.V_2 := V_1 \otimes V_2 \bigoplus V_2 \otimes V_1 \subseteq S^2(V).$$

Another notation that we are going to use is the map  $\mu : S^2(\mathbb{C}^{\underline{N}}) \longrightarrow \mathbb{C}^{2\underline{N}}$ . Given two multi-vectors  $f, g \in \mathbb{C}^{\underline{N}}$ , let  $\mu(f.g)$  be the multi-vector in  $\mathbb{C}^{2\underline{N}}$  defined by

$$\mu(f.g)_\beta = \sum_{\alpha_1 + \alpha_2 = \beta} f_{\alpha_1} g_{\alpha_2}.$$

We will call  $\mu$  the *polynomial multiplication* or the *convolution product*. This is because an element  $f \in \mathbb{C}^{\underline{N}}$  may be considered as a polynomial in  $r$  variables of degree  $N$ ; the coefficient of the monomial  $\alpha = (i_1, \dots, i_r) \in \underline{N}$  is  $f(\alpha)$ , then for  $f, g \in \mathbb{C}^{\underline{N}}$ , we may consider the product  $\mu(f.g) \in \mathbb{C}^{2\underline{N}}$ .

The following Proposition gives an isomorphism between  $\phi_A(\text{Cat})$  and  $\mathbb{P}(\mu(\text{im}A^t.\text{im}A^t)^\vee)$ . This isomorphism is the restriction of the following one:

**Lemma 8.** *There exist an isomorphism between the projective space of catalectic multi-matrixes,  $\text{Cat}$ , and the dual of the image of  $\mu : S^2(\mathbb{C}^{\underline{N}}) \longrightarrow \mathbb{C}^{2\underline{N}}$ ,*

$$\text{Cat} \longrightarrow \mathbb{P}\mu(S^2(\mathbb{C}^{\underline{N}}))^\vee, \quad B \rightarrow \widehat{b^t}.$$

where  $b \in \mathbb{C}^{2\underline{N}}$  is a multi-vector that defines  $B$  and the associated linear functional  $\widehat{b^t} : \mathbb{C}^{2\underline{N}} \rightarrow \mathbb{C}$  is  $\widehat{b^t}(x) = b^t x$ .

Even more,  $B : S^2(\mathbb{C}^{\underline{N}}) \rightarrow \mathbb{C}$  as a symmetric form,  $x^t B y$ , is equal to the symmetric form  $\widehat{b^t} \circ \mu : S^2(\mathbb{C}^{\underline{N}}) \rightarrow \mathbb{C}$  given by  $\widehat{b^t}(\mu(x.y))$ .

**PROOF.** First of all, let's see that the map  $\mu$  is surjective. Let  $x_i$  be the multi-vector that has a 1 in the  $i$ -th place and 0 in the rest. The multi-vectors  $\{x_i\}_{i \leq 2N}$  generate  $\mathbb{C}^{2\underline{N}}$  and also,  $x_i \in \text{im}(\mu)$ . Then  $\mu$  is surjective.

The definition of a catalectic multi-matrix  $B$  implies that there exist a multi-vector  $b \in \mathbb{C}^{2\underline{N}}$  such that  $B_{ij} = b_{i+j}$ . It is easy to see that this multi-vector  $b$  is unique, suppose  $b, b'$  defines the same catalectic multi-matrix, then

$$b_i = B_{i0} = b'_i, \quad b_{N+j} = B_{Nj} = b'_{N+j}, \quad 0 \leq i, j \leq N \implies b = b'.$$

Note that the multi-vector  $b$  is the concatenation of the 0-row and  $N$ -column of  $B$ .

Finally, we have defined a linear isomorphism  $B \rightarrow \widehat{b^t}$  and also,

$$x_i^t B x_j = B_{ij} = b_{i+j} = \widehat{b^t}(\mu(x_i.x_j)).$$

□

**Proposition 9.** *Let  $\mu : S^2(\mathbb{C}^{\underline{N}}) \longrightarrow \mathbb{C}^{2\underline{N}}$  be the polynomial multiplication. Let  $A \in \mathbb{C}^{\ell \times \underline{N}}$ , then*

$$\phi_A(\text{Cat}) \longrightarrow \mathbb{P}\mu(\text{im}A^t.\text{im}A^t)^\vee, \quad ABA^t \rightarrow \widehat{b^t}|_{\mu(\text{im}A^t.\text{im}A^t)}.$$

*is a linear isomorphism. We will identify multi-matrixes  $ABA^t$  with functionals on  $\mu(\text{im}A^t.\text{im}A^t)$ .*

PROOF. Let  $B$  be a catalectic multi-matrix,  $b \in \mathbb{C}^{2N}$  its associated multi-vector and let  $A \in \mathbb{C}^{\ell \times N}$ . Let  $x_i$  be the multi-vector that has a 1 in the  $i$ -th place and a 0 in the rest, then

$$(ABA^t)_{ij} = x_i^t ABA^t x_j = (x_i A^t)^t B(A^t x_j) = \widehat{b^t}(\mu(A^t x_i, A^t x_j)).$$

Let's see that the following linear map is well defined and has an inverse,

$$\begin{aligned} \phi_A(Cat) &\longrightarrow \mathbb{P}(\mu(\text{im} A^t, \text{im} A^t)^\vee), \\ ABA^t &\longrightarrow \widehat{b^t}|_{\mu(\text{im} A^t, \text{im} A^t)}. \end{aligned}$$

Suppose  $AB_1 A^t = AB_2 A^t$  then

$$\widehat{b_1^t}(\mu(A^t x_i, A^t x_j)) = (AB_1 A^t)_{ij} = (AB_2 A^t)_{ij} = \widehat{b_2^t}(\mu(A^t x_i, A^t x_j)).$$

Suppose  $\widehat{b_1^t}|_{\mu(\text{im} A^t, \text{im} A^t)} = \widehat{b_2^t}|_{\mu(\text{im} A^t, \text{im} A^t)}$  then

$$(AB_1 A^t)_{ij} = \widehat{b_1^t}(\mu(A^t x_i, A^t x_j)) = \widehat{b_2^t}(\mu(A^t x_i, A^t x_j)) = (AB_2 A^t)_{ij}.$$

□

By now we have the isomorphism  $\phi_A(Cat) \cong \mathbb{P}(\mu(\text{im} A^t, \text{im} A^t)^\vee)$ , but we need to characterize those linear functionals that corresponds to multi-matrixes  $\langle ABA^t \rangle \in \mathcal{V}$ . In the next Theorem, we will prove that the following map parameterize them.

**Lemma 10.** *Let  $A \in \mathbb{C}^{\ell \times N}$  then the following map is well defined*

$$\Psi : Gr^1(\text{im} A^t) \rightarrow \mathbb{P}(\mu(\text{im} A^t, \text{im} A^t)^\vee), \quad W \rightarrow \mu(W, \text{im} A^t)^\circ = \langle \phi \rangle,$$

where  $Gr^1(\text{im} A^t)$  is the variety of hyperplanes in  $\text{im} A^t$  and the symbol  $\circ$  is the annihilator of vector spaces  $\mu(W, \text{im} A^t) \subseteq \mu(\text{im} A^t, \text{im} A^t)$ , that is,  $\ker \phi = \mu(W, \text{im} A^t)$ .

PROOF. Let's see first that the map

$$Gr^1(\text{im} A^t) \rightarrow \mathbb{P}(\mu(\text{im} A^t, \text{im} A^t)^\vee), \quad W \rightarrow \mu(W, \text{im} A^t)^\circ = \langle \phi \rangle$$

is well defined. Let  $W \in Gr^1(\text{im} A^t)$  and consider the following short exact sequence

$$0 \longrightarrow \ker \mu \cap (W, \text{im} A^t) \longrightarrow W, \text{im} A^t \xrightarrow{\mu} \mu(W, \text{im} A^t) \longrightarrow 0.$$

Let  $K := \ker \mu \cap (\text{im} A^t, \text{im} A^t)$  and given that  $W \subseteq \text{im} A^t$  we have

$$\ker \mu \cap (W, \text{im} A^t) = \ker \mu \cap (W, \text{im} A^t) \cap (\text{im} A^t, \text{im} A^t) = K \cap (W, \text{im} A^t).$$

Let's see that in fact  $K \subseteq W, \text{im} A^t$ . Let  $v \in \text{im} A^t$  be such that  $\langle v \rangle \oplus W = \text{im} A^t$ , then  $\langle v, v \rangle \oplus W, \text{im} A^t = \text{im} A^t, \text{im} A^t$ . Given that  $v \neq 0$  we have

$$\mu(v, v) \neq 0 \implies v, v \notin K \implies \langle v, v \rangle \cap K = 0 \implies K \subseteq W, \text{im} A^t.$$

In other words, for any  $W \in Gr^1(\text{im} A^t)$  we have  $K \subseteq W, \text{im} A^t$  then the following two short exact sequences have the same kernel  $K$ ,

$$0 \longrightarrow K \longrightarrow W, \text{im} A^t \xrightarrow{\mu} \mu(W, \text{im} A^t) \longrightarrow 0,$$



$$0 \longrightarrow K \longrightarrow \text{im}A^t.\text{im}A^t \xrightarrow{\mu} \mu(\text{im}A^t.\text{im}A^t) \longrightarrow 0,$$

then  $\mu(W.\text{im}A^t)$  is a hyperplane of  $\mu(\text{im}A^t.\text{im}A^t)$ , so the following morphism is well defined,

$$Gr^1(\text{im}A^t) \longrightarrow Gr^1(\mu(\text{im}A^t.\text{im}A^t)), \quad W \rightarrow \mu(W.\text{im}A^t).$$

Identifying  $Gr^1(\mu(\text{im}A^t.\text{im}A^t))$  with  $\mathbb{P}\mu(\text{im}A^t.\text{im}A^t)^\vee$ , for every hyperplane  $W \subseteq \text{im}A^t$ , there exist a functional  $\phi : \mu(\text{im}A^t.\text{im}A^t) \longrightarrow \mathbb{C}$  such that  $\mu(W.\text{im}A^t) = \ker \phi$ , specifically,

$$Gr^1(\text{im}A^t) \rightarrow \mathbb{P}\mu(\text{im}A^t.\text{im}A^t)^\vee, \quad W \rightarrow \mu(W.\text{im}A^t)^\circ = \langle \phi \rangle.$$

□

We are now in the position to prove the irreducibility of  $M_y$ ,

**Theorem 11.** *Let  $A \in \mathbb{C}^{t \times N}$  then  $\phi_A(\text{Cat}) \cap \mathcal{V}$  is irreducible. As a corollary, given  $y \in V$  there exist a multi-matrix  $A$  such that  $M_y \cong \phi_A(\text{Cat}) \cap \mathcal{V}$  (see 6), then the variety  $M_y$  is irreducible.*

PROOF. Let's see first that the image of  $\Psi$  corresponds to multi-matrixes  $ABA^t \in \mathcal{V}$  (for the definition of  $\Psi$  see 10).

From 9 we know that the multi-matrix  $ABA^t$  has associated a functional  $\widehat{b^t}|_{\mu(\text{im}A^t.\text{im}A^t)}$ . Even more  $\text{rk}(ABA^t) \leq 1$  if and only if there exist a codimension one hyperplane  $W \subseteq \text{im}A^t$  such that  $x'ABA^ty = 0$  for all  $A^ty \in W$  and for all  $A^tx \in \text{im}A^t$ . This is equivalent to  $\widehat{b^t}(\mu(A^tx.A^ty)) = 0$  for all  $A^ty \in W$  and for all  $A^tx \in \text{im}A^t$ , i.e.

$$\text{rk}(ABA^t) \leq 1 \iff \exists W \in Gr^1(\text{im}A^t) \mid \mu(W.\text{im}A^t) \subseteq \ker(\widehat{b^t}|_{\mu(\text{im}A^t.\text{im}A^t)}),$$

where  $Gr^1(\text{im}A^t)$  is the variety of hyperplanes in  $\text{im}A^t$ .

We know from 10 that  $\mu(W.\text{im}A^t)$  is an hyperplane of  $\mu(\text{im}A^t.\text{im}A^t)$ , and then the kernel of  $\widehat{b^t}|_{\mu(\text{im}A^t.\text{im}A^t)}$  must be equal to  $\mu(W.\text{im}A^t)$ . In other words, the image of  $\Psi$  corresponds to those functional whose kernel are equal to  $\mu(W.\text{im}A^t)$  for some  $W \in Gr^1(\text{im}A^t)$ , equivalently, the image of  $\Psi$  corresponds to multi-matrixes  $ABA^t \in \mathcal{V}$ .

Summing up, we have the following isomorphisms that implies the irreducibility

$$\begin{aligned} \phi_A(\text{Cat}) \cap \mathcal{V} &= \{\langle ABA^t \rangle \mid \text{rk}(ABA^t) \leq 1\} = \\ &= \{\langle ABA^t \rangle \mid \exists W \in Gr^1(\text{im}A^t), \mu(W.\text{im}A^t) \subseteq \ker \widehat{b^t}\} \cong \\ &= \{\langle \widehat{b^t}|_{\mu(\text{im}A^t.\text{im}A^t)} \rangle \mid \exists W \in Gr^1(\text{im}A^t), \mu(W.\text{im}A^t) = \ker \widehat{b^t}|_{\mu(\text{im}A^t.\text{im}A^t)}\} = \\ &= \{\mu(W.\text{im}A^t)^\circ \in \mathbb{P}\mu(\text{im}A^t.\text{im}A^t)^\vee \mid W \in Gr^1(\text{im}A^t)\}. \end{aligned}$$

Let's characterize the last variety. Consider an inner product in  $\text{im}A^t$  and for every  $W \in Gr^1(\text{im}A^t)$  let  $\langle v \rangle = W^\perp$ . From the proof of 10 it is easy to see that  $\mu(v.v) \oplus \mu(W.\text{im}A^t) = \mu(\text{im}A^t.\text{im}A^t)$ , then

$$\{\mu(W.\text{im}A^t)^\circ \mid W \in Gr^1(\text{im}A^t)\} \cong \{\langle \mu(v.v) \rangle \mid v \in \text{im}A^t\} \subseteq \mathbb{P}\mu(\text{im}A^t.\text{im}A^t).$$

□

#### 4. Applications.

**Corollary 12.** *Let  $V$  be a representation of a simple Lie algebra  $\mathfrak{g}$ . Let  $G$  be a simply connected Lie group with Lie algebra  $\mathfrak{g}$  and let  $X \subseteq \mathbb{P}V$  be a variety stable under  $G$  with a dense orbit  $G.y$ . Then  $M_y$  is the intersection of quadrics that contains  $X$ ,  $M_y$  is an irreducible variety and  $I(X)_2 = I(M_y)_2$ .*

PROOF. Follows from the fact that the smallest  $\mathfrak{g}$ -module that contains  $yy \in S^2(V)$  is the same as the smallest  $G$ -module that contains  $yy \in S^2(V)$ , that is,  $U\mathfrak{g}(yy) = \langle G.yy \rangle$ . Then

$$I(X)_2 = \{q \mid q(x) = 0, x \in X\} = \{q \mid q(g.y) = 0, g \in G\} =$$

$$\{q \mid q(\langle G.yy \rangle) = 0\} = \{q \mid q(U\mathfrak{g}(yy)) = 0\} = U\mathfrak{g}(yy)^\circ = I(M_y)_2.$$

Recall that  $M_y$  is generated in degree two. □

**Remark 13.** In [2, 1.3.29] there is a sufficient condition for a variety to have a dense orbit. It says that when the action of  $G$  in  $V$  has a finite number of orbits, any irreducible  $G$ -stable variety  $X \subseteq \mathbb{P}V$  is the closure of an orbit  $G.y$ .

In the next Theorem we will give another result that guaranties that the base-locus of quadrics containing a variety is irreducible. The hypothesis is over the module  $V$  independently of the varieties. But first we will need a Lemma:

**Lemma 14.** *Let  $W$  be a  $\mathfrak{g}$ -module and let  $w = w_1 + \dots + w_k \in W$ , with  $w_i \in W_i$ ,  $w_i \neq 0$  and  $W_i$  a simple submodule of  $W$  ( $1 \leq i \leq k$ ). Suppose that  $W_i \not\cong W_j$  for  $i \neq j$  then*

$$U\mathfrak{g} w = W_1 \oplus \dots \oplus W_k$$

PROOF. Let  $p_i$  be a maximal weight vector of  $W_i$  of weight  $\omega_i$  ( $1 \leq i \leq k$ ). Given that  $W_i \not\cong W_j$  the weights  $\omega_i \in \mathfrak{h}^\vee$  are all different ([7, p.58]).

Case one: Assume that  $w = p_1 + \dots + p_k$  is a sum of maximal weight vectors. Given that they are all different, there exist  $P \in U\mathfrak{g}$  such that  $Pw = Pp_i \neq 0$  for some  $1 \leq i \leq k$ . On the other hand, given that  $Pp_i \neq 0$ , it generates the whole submodule  $W_i$  and then there exist  $Q \in U\mathfrak{g}$  such that  $QPw = p_i$ . Finally we proceed by induction for  $w - p_i$ .

Case two: If  $w = w_1 + \dots + w_k$  then there exist  $P \in U\mathfrak{g}$  such that  $Pw$  is a sum of maximal weight vectors. Then apply case one. □

**Theorem 15.** *Let  $V$  be a  $G$ -module such that  $S^2(V) = W_1 \oplus \dots \oplus W_k$ ,  $W_i \not\cong W_j$ . Let  $X \subseteq \mathbb{P}V$  be an irreducible  $G$ -stable variety. Then there exists a generic  $y \in X$  such that*

$$M_y = \{\langle x \rangle \in \mathbb{P}V \mid q(x) = 0 \forall q \in I(X)_2\}.$$

*In other words, the intersection of the quadrics that contains  $X$  is an irreducible variety.*

PROOF. Let  $C \subseteq V$  be the irreducible cone associated to  $X \subseteq \mathbb{P}V$ . Let  $S_X$  be the smallest submodule of  $S^2(V)$  that contains  $\{cc \mid c \in C\}$ . Given  $W_i \subseteq S_X$ , let  $\pi_i : S^2(V) \rightarrow W_i$  be the projection to  $W_i$  and

$$H_i := \{\pi_i = 0\} = \ker \pi_i.$$

Note that  $S_X \not\subseteq H_i$  and given that  $H_i$  is a module, we have  $\{cc \mid c \in C\} \not\subseteq H_i$ .

Let  $H := \bigcup_i H_i$ , then  $\{cc \mid c \in C\} \setminus H$  is a Zariski dense subset of  $\{cc \mid c \in C\}$ . Then there exist a generic  $yy \notin H$  such that  $y \in C$ .

$$yy = \sum a_i w_i \quad a_i = \pi_i(yy) \neq 0 \implies U\mathfrak{g}(yy) = S_X.$$

The last implication follows from 14. Finally  $I(X)_2 = S_X^\circ = U\mathfrak{g}(yy)^\circ = I(M_y)_2$ .  $\square$

**Remark 16.** With this Theorem we can answer the question of the exercise [4, 15.44 Hard Exercise] (see the Introduction of this paper). Using the fact that  $S^2(\wedge^k V)$  has a decomposition into non-isomorphic simple submodules,

$$S^2(\wedge^k V) = \bigoplus_{i \geq 0} \Theta_{2i},$$

and that the  $p$ -restricted chordal variety  $C^p(Gr^k(V))$  is irreducible we can say that the intersection of all the quadrics that contains  $C^p(Gr^k(V))$  is an irreducible variety.

**Corollary 17.** Let  $V$  be a  $G$ -module such that  $S^2(V) = W_1 \oplus \dots \oplus W_k$ ,  $W_i \not\cong W_j$ . Let  $X \subseteq \mathbb{P}V$  be a  $G$ -stable variety defined by quadrics. Then there exists  $x_1, \dots, x_s \in X$  such that the irreducible components of  $X$  are of the form  $X = M_{x_1} \cup \dots \cup M_{x_s}$ .

PROOF. Let  $X = X_1 \cup \dots \cup X_s$  be the decomposition of  $X$  into irreducible components. Let  $x_1 \in X_1$  be a generic element and consider the irreducible variety  $M_{x_1}$  defined by  $I(X_1)_2$ , then

$$I(M_{x_1})_2 = I(X_1)_2 \supseteq I(X)_2$$

Given that  $M_{x_1}$  and  $X$  are defined by quadrics,  $M_{x_1} \subseteq X$ , also,  $X_1 \subseteq M_{x_1}$ . Being  $M_{x_1}$  irreducible, we have  $M_{x_1} = X_1$ . Repeat this for the remaining components  $X_i$ ,  $2 \leq i \leq s$ .  $\square$

**Corollary 18.** Let  $V$  be a  $G$ -module such that  $S^2(V^\vee) = W_1 \oplus \dots \oplus W_k$ ,  $W_i \not\cong W_j$ . Let  $X \subseteq \mathbb{P}V$  be a  $G$ -stable variety defined by

$$I(X)_2 = W_2 \oplus \dots \oplus W_k$$

then  $X$  is irreducible. Even more, if the ideal in degree two is

$$I(X)_2 = W_{s+1} \oplus \dots \oplus W_k$$

then  $X$  has at most  $\varphi(s)$  irreducible components (it could be irreducible like in 16, or even empty). The set function  $\varphi(s)$  count the maximum number of subsets  $\{S_1, \dots, S_{\varphi(s)}\}$  of a set of  $s$  elements such that  $S_i \not\subseteq S_j$ . We have  $\varphi(1) = 1, \varphi(2) = 2, \varphi(3) = 3, \varphi(4) = 6$ .

PROOF. First note that  $S^2(V^\vee)$  has all the simple submodules non-isomorphic if and only if  $S^2(V)$  has all the simple submodules non-isomorphic. Assume now that  $I(X)_2 = W_2 \oplus \dots \oplus W_k$  then the ideal in degree two of an irreducible component  $M_{x_1}$  contains  $I(X)_2$ ,

$$I(X)_2 \subseteq I(M_{x_1})_2,$$

then the simple module  $W_1$  is in  $I(M_{x_1})_2$  or not. In both cases  $X$  is irreducible.

Assume now that  $I(X)_2 = W_{k+1} \oplus \dots \oplus W_k$ . Let  $X = M_{x_1} \cup \dots \cup M_{x_r}$  be the irreducible decomposition of  $X$ . The simple submodules of  $I(M_{x_i})_2$  that are not contained in  $I(X)_2$  determine a subset  $S_i \subseteq \{1, \dots, s\}$ . Note that  $M_{x_i} \not\subseteq M_{x_j}$  if and only if  $S_i \not\subseteq S_j$ .  $\square$

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